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Analytical Bethe Ansatz, Canonical Bäcklund Transformation and Q-Operator For A New Discrete Integrable Heirarchy

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A new discrete heiarchy of integrable equations is generated from a new Lax Operator and a canonical Bäcklund transformation of the system is derived using Sklyanin's formalism, based on the classical r-matrix. By quantising the system a quantum analogue of the corresponding canonical Bäcklund transformation is obtained and certain properties of the associated Q-operator are examined. Finally the analytical Bethe Ansatz is used to solve for the spectrum.

KEY WORDS: Bethe ansatz; canonical backlund transformation; Q-operator.

1. INTRODUCTION

In the study of completely integrable systems, a special place is held by models which are discrete in nature, owing to the existence of concrete methods for their quantization (Korepin *et al.*, 1993). An elegant feature of any integrable system is its proliferation of its solutions obtained by means of a Bäcklund transformation (Sklyanin, 1999). Recently a new approach for studying Bäcklund transformation was initiated by Sklyanin *et al* in case of discrete integrable systems. The Bäcklund transformation (CBT) derived by this method had its additional feature of being canonical transformations, by the very nature of their construction: being derivable from a suitable generating function. Further more their canonical nature enables one to continue to the quantum domain and it is possible to deduce the Baxter's Q operator which represents in some sense the quantum counterpart of the classical Canonical Bäcklund transformation . In the quantum regime it is legitimate exercise to explore the excitation spectrum using the appropriate version of

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the Bethe Ansatz. On closer examination it is found that the entire procedure for obtaining such canonical Bäcklund transformations is an outcome of the theory of classical r-matrix theory first formulated by Semenov-Tian-Shansky (in press).

Over the last couple of decades, the quantum inverse scattering method has become a standard technique for determining the exact eigen values and Bethe states for a wide class of integrable models mostly by means of algebraic Bethe Ansatz. However quantization of Bäcklund transformation is a much more recent notion. Indeed Skylanin's elegant method of deriving canonical Bäcklund transformations is crucially dependent on finding two different representations of classical r-matrix equation

$$\{L(x,\lambda) \overset{\otimes}{,} L(y,\mu)\} = [r(\lambda-\mu), L(x,\lambda) \otimes L(y,\mu)]\delta(x-y)$$

as will be evident in Section 3.

In this communication we have introduced a new Lax operator and have derived a new set of discrete nonlinear integrable equations in Section 2. The canonical dynamical variables may be identified through the trace identity technique of Zhang *et al.* (1991). In Section 3 we have obtained the canonical Bäcklund transformation for the system and its generating function. In Section 4 we have breifly outlined the construction of the corresponding quantum mechanical counterpart and have obtained the Q operator. Finally, in Section 5 by using the analytical Bethe Ansatz we have obtained excitation spectrum.

2. FORMULATION

We consider the following discrete Lax operator defined at the nth lattice site by

$$L_n = \begin{pmatrix} 0 & -x_n^{-1} \\ x_n & u + ip_n x_n \end{pmatrix} \tag{2.1}$$

and satisfying the corresponding lax equation

$$\psi_{n+1} = l_n \psi_n \tag{2.2}$$

where u is the complex spectral parameter. The time evolution of ψ_n is given by

$$\psi_{n,t} = V_n \psi_n. \tag{2.3}$$

In Eq. (2.1) x_n and p_n are the dynamical variables satisfying the poisson brackets

$$\{p_n, x_m\} = \delta_{nm}, \{p_n, p_m\} = \{x_n, x_m\} = 0. \tag{2.4}$$

We begin by writing the time evolution matrix

$$V_n = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} \tag{2.5}$$

with the usual expansions of A_n, B_n etc. in the following form:

$$A_n = \sum_{i=-m}^{m} u^i a_i^n, \quad B_n = \sum_{i=-m}^{m} u^i b_i^n. \tag{2.6}$$

For example we consider here the simplest case when $m = 1$, so that

$$\begin{aligned} A_n(u) &= \frac{1}{u} a_{-1}^n + a_0^n + u a_1^n \\ B_n(u) &= \frac{1}{u} b_{-1}^n + b_0^n + u b_1^n \\ C_n(u) &= \frac{1}{u} c_{-1}^n + c_0^n + u c_1^n \\ D_n(u) &= \frac{1}{u} d_{-1}^n + d_0^n + u d_1^n. \end{aligned} \tag{2.7}$$

Then the compatibility condition of (2.2) and (2.3) leads to the following coupled nonlinear system

$$\begin{aligned} x_{n,t} &= -(a'' - d'')x_n - i(a' - d')p_n^{-1} + i p_n x_n^2 (a - d) \\ p_{n,t} &= -i(a - d)\left(x_n^{-2} x_{n+1} - x_{n-1}^{-1}\right) + p_n(a'' - d'') \\ &\quad + i x_n^{-1}(a' - d') - i x_n p_n^2 (a - d) \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} a_0^n &= a'' = \text{constant}, \quad a_1^n = a = \text{constant}, \quad a_{-1}^n = a' = \text{constant} \\ d_0^n &= d'' = \text{constant}, \quad d_1^n = d = \text{constant}, \quad d_{-1}^n = d' = \text{constant} \\ b_0^n &= x_{n-1}^{-1}(a - d) \quad b_1^n = 0 \quad b_{-1}^n = -i x_{n-1}^{-2} p_{n-1}^{-1}(a' - d') \\ c_0^n &= x_n(d - a) \quad c_1^n = 0 \quad c_{-1}^n = i p_n^{-1}(a' - d'). \end{aligned} \tag{2.9}$$

If we set

$$\begin{aligned} (a'' - d'') &= \alpha \equiv -1 \\ (a' - d') &= \beta \equiv +i \\ (a - d) &= \gamma \equiv +i \end{aligned}$$

then (2.8) may be expressed in the form

$$x_{n,t} = x_n(1 - p_n x_n) + p_n^{-1}$$
$$p_{n,t} = -p_n(1 - p_n x_n) + \left(\frac{x_{n+1} - x_n}{x_n^2} - \frac{1}{x_{n-1}}\right) \quad (2.10)$$

It may now be easily verified that the symplectic structure deduced from the trace identity formulae of Zhang *et al.* actually leads to the Poisson brackets (2.4).

Having derived the new hiearchy of discrete nonlinear equations (2.8), we next proceed to study one of its most interesting properties—namely the generation of new solutions from existing ones.

3. BÄCKLUND TRANSFORMATION

The Lax operator given by Eq. (2.1) together with the basic poisson brackets (2.4) immediately leads to the following fundamental poisson brackets for the Lax operator, viz.

$$\{L_n(u) \overset{\otimes}{,} L_m(v)\} = [r(u, v), L_n(u) \otimes L_m(v)]\delta_{nm} \quad (3.1)$$

where $r(u, v)$ represents the classical r- matrix and is given by

$$r(u, v) = \frac{iP}{u - v},$$

with P being the permutation operator satisfying $P(x \otimes y) = y \otimes x$. In order to find a local canonical Bäcklund transformation for the above system, i.e

$$B_k : (p_k, x_k) \longrightarrow (\tilde{p}_k, \tilde{x}_k). \quad (3.2)$$

We employ the technique devised by Sklyanin in Sklyanin (1999). We assume that the transformation depends on an arbitrary complex parameter λ. Owing to the canonical nature of the transformation the Hamiltonians and associated Poisson structure should remain unchanged when expressed in terms of the new variables $(\tilde{p}_k, \tilde{x}_k)$. As stated earlier the main requirement is to find another Lax operator $M_n(u)$ possessing the same r matrix and satisfying the relation (3.1). The point here is that of the several matrices obeying the r-matrix relation (3.1) one has to choose a particular one which satisfies the following :

$$M_k(u, \lambda)\, l_k(p_k, x_k; u) = \tilde{l}_k(\tilde{p}_k, \tilde{x}_k; u)\, M_k(u, \lambda), \quad k = 1 \ldots N \quad (3.3)$$

where N represents the total number of lattice sites. Relation (3.3) is assumed to hold for the values of the spectral parameter u. To account for the dynamical variables involved in the definition of $M_k(u; \lambda)$, Sklyanin formulated an elegant method based on the concept of an extended phase space. That is, besides the basic variables (p_k, x_k) and $(\tilde{p}_k, \tilde{x}_k)$, one needs to consider additional phase space variables (q_k, r_k) and (s_k, t_k), known as auxiliary phase space variables required to define $M_k(u, \lambda)$; so that Eq. (3.3) should be written as

$$M_k(u, \lambda; q_k, r_k)\; l_k(u; p_k, x_k) = \tilde{l}_k(u; \tilde{p}_k, \tilde{x}_k)\; M_k(u, \lambda; s_k, t_k) \tag{3.4}$$

with (q_k, r_j) and (s_k, t_j) satisfying the poisson brackets

$$\begin{aligned} &\{q_k, r_j\} = \delta_{kj}, \quad \{s_k, t_j\} = \delta_{kj}. \\ &\{q_k, q_j\} = 0 = \{r_k, r_j\}, \quad \{s_k, s_j\} = 0 = \{t_k, t_j\}. \end{aligned} \tag{3.5}$$

Since the local Bäcklund transformation of B_k of Eq. (3.2) is assumed to be a mapping from (p_k, x_k) to $(\tilde{p}_k, \tilde{x}_k)$, the auxiliary variables (q_k, r_k) and (s_k, t_k) involved in Eq. (3.4) should ultimately be eliminated. To this end we shall impose the following constraints

$$s_k = q_{k-1}, \quad t_k = r_{k-1}. \tag{3.6}$$

Regarding the particular choice of $M_k(u, \lambda)$ we consider the Lax operator

$$M_k(u, \lambda; q_k, r_k) \equiv \begin{pmatrix} u - \lambda + q_k r_k & q_k \\ r_k & 1 \end{pmatrix}. \tag{3.7}$$

Substituting (3.7) into (3.4) and equating coefficients of the different powers of the spectral parameter u, leads to the following set of equation:

$$\begin{aligned} q_k r_k &= -\tilde{x}_k^{-1} t_k, \quad -\tilde{x}_k = t_k, \quad q_k = x_k^{-1}. \\ x_k &= \tilde{x}_k(s_k t_k - \lambda) + i t_k \tilde{p}_k \tilde{x}_k \\ \tilde{x}_k s_k + i \tilde{p}_k \tilde{x}_k &= -r_k x_k^{-1} + i p_k x_k \\ -\tilde{x}_k^{-1} &= x_k^{-1}(\lambda - q_k r_k) + i q_k p_k x_k \end{aligned} \tag{3.8}$$

We solve the above set of Eqs. (3.8) for the following variables.

$$\begin{aligned} t_k &= -\tilde{x}_k, \quad q_k = x_k^{-1} \\ i p_k &= -\left(\frac{\lambda}{x_k} + \frac{1}{\tilde{x}_k}\right) + \frac{r_k}{x_k^2} \\ i \tilde{p}_k &= -\left(\frac{\lambda}{\tilde{x}_k} + \frac{x_k}{x_k^2} + s_k\right. . \end{aligned} \tag{3.9}$$

Upon imposing the constraints stated in Eq. (3.6) in Eq. (3.9) we obtain

$$\begin{aligned} i p_k &= -\left(\frac{\lambda}{x_k} + \frac{1}{\tilde{x}_k} + \frac{\tilde{x}_{k+1}}{x_k^2}\right) \\ \tilde{p}_k &= \frac{\lambda}{\tilde{x}_k} + \frac{1}{x_{k-1}} + \frac{x_k}{\tilde{x}_k^2} \end{aligned} \tag{3.10}$$

Equation (3.10) represents the requisite local Bäcklund transformation between the variables (p_k, x_k) and $(\tilde{p}_k, \tilde{x}_k)$. Furthermore, it may be easily verified that these equations may be derived from a generating function $F(\lambda; \{x_k\}, \{\tilde{x}_k\})$ i.e.;

$$p_k = \frac{\partial F}{\partial x_k}$$
$$\tilde{p}_k = -\frac{\partial F}{\partial \tilde{x}_k} \tag{3.11}$$

where

$$F(\lambda; \{x_k\}, \{\tilde{x}_k\}) = i \sum_{k=1}^{N} \left\{ -\frac{\tilde{x}_{k+1}}{x_k} + \frac{x_k}{\tilde{x}_k} + \lambda \ln \frac{x_k}{\tilde{x}_k} \right\}. \tag{3.12}$$

4. CONSTRUCTION OF Q-OPERATOR AND ITS PROPERTIES

An interesting aspect of the quantum mechanical consideration of integrable systems, which has emerged in recent times, is that Baxter's Q-operator, which was originally deduced in the context of lattice models in solid state physics, is nothing but the quantum version of a canonical Bäcklund transformation. The constructon of Q-operator is based on the idea of finding an integral operator $\hat{R}_\lambda$ defined by

$$\hat{R}_\lambda \phi(x, q) = \int dx \int dq R_\lambda(\tilde{x}, s | x, q) \phi(x, q). \tag{4.1}$$

Recalling the fact the canonical Bäcklund transformation discussed in the preceeding section, represents a mapping from the variables $(p_k, x_k) \longrightarrow (\tilde{p}_k, \tilde{x}_k)$, the first step in the construction of the Q-operator or more specifically $\hat{R}_\lambda$ consists in representing the Lax operator $l_k(u)$ of Eq. (2.1) in terms of differential operators satisfying the basic quantum mechanical commutation rules

$$[p_k, x_j] = -i\eta\delta_{kj}, \quad [\tilde{p}_k, \tilde{x}_j] = -i\eta\delta_{kj}. \tag{4.2}$$

Here we adopt the following representations of p_k and $\tilde{p}_k$ namely

$$p_k = -i\eta \frac{\partial}{\partial x_k}, \quad \tilde{p}_k = -i\eta \frac{\partial}{\partial \tilde{x}_k} \tag{4.3}$$

such that the local quantum mechanical Lax operator $l_k(u)$ assumes the following form

$$l_k(p_k, x_k, u) = l_k\left(-i\eta \frac{\partial}{\partial x_k}, x_k, u\right) = \begin{pmatrix} 0 & -x_k^{-1} \\ x_k & u + \eta \frac{\partial}{\partial x_k} x_k \end{pmatrix} \tag{4.4}$$

while the matrix $M_k(q, r, u - \lambda)$ becomes

$$M_k(u-\lambda) = \begin{pmatrix} u - \lambda + \eta q_k \frac{\partial}{\partial q_k} & q_k \\ \eta q_k & 1 \end{pmatrix}. \tag{4.5}$$

The crucial point in the determination of any Q-operator is that it commutes with the trace t(u) of the monodromy matrix $T(u) \equiv \prod_{k=1}^{N} l_k(u)$. This will be true if the kernel of the operator $\hat{R}_\lambda$ viz R_λ satisfies the following relation (Kuznetsov *et al.*, 2000)

$$R_\lambda M\left(u-\lambda, q, \frac{\partial}{\partial q}\right) l\left(u, x, \frac{\partial}{\partial x}\right. = \tilde{l}\left(u, \tilde{x}, \frac{\partial}{\partial \tilde{x}}\right) M\left(u-\lambda, s, \frac{\partial}{\partial s}\right) R_\lambda \tag{4.6}$$

where for notational convenience we have omitted the lattice site index "k." Consequently we have the following relation:

$$R_\lambda \begin{pmatrix} u - \lambda + \eta q \frac{\partial}{\partial q} & q \\ \eta \frac{\partial}{\partial q} & 1 \end{pmatrix} \begin{pmatrix} 0 & -x^{-1} \\ x & u + \eta \frac{\partial}{\partial x} x \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -\tilde{x}^{-1} \\ \tilde{x} & u + \eta \frac{\partial}{\partial \tilde{x}} \tilde{x} \end{pmatrix} \begin{pmatrix} u - \lambda + \eta s \frac{\partial}{\partial s} & s \\ \eta \frac{\partial}{\partial s} & 1 \end{pmatrix} R_\lambda. \tag{4.7}$$

Utilising the properties of adjoint operators we can shift R_λ occuring on the left hand side of (4.7) to the right, whence upon equating the coefficients of he powers of u, we are led to the following set of equations:

$$\frac{\partial R_\lambda}{\partial s} = -\frac{qx}{\eta \tilde{x}^{-1}} R_\lambda \tag{4.8}$$

$$(q - x^{-1}) R_\lambda = 0 \tag{4.9}$$

$$\eta q \left(x \frac{\partial R_\lambda}{\partial x} - x^{-1} \frac{\partial R_\lambda}{\partial q} \right) = \{(\lambda + \eta) x^{-1} + \tilde{x}^{-1}\} R_\lambda \tag{4.10}$$

$$(x + \lambda \tilde{x}) R_\lambda = \eta(\tilde{x} s + \eta) \frac{\partial R_\lambda}{\partial s} + \eta^2 \tilde{x} \frac{\partial^2 R_\lambda}{\partial s \partial \eta} \tag{4.11}$$

$$x^{-1} \frac{\partial R_\lambda}{\partial q} - x \frac{\partial R_\lambda}{\partial x} - y \frac{\partial R_\lambda}{\partial y} = \left(\frac{\tilde{x} s}{\eta} + 1 \right) R_\lambda. \tag{4.12}$$

Upon solving the above set of partial differential equations we arrive at the following solution for the kernel R_λ:

$$R_\lambda(\tilde{x}, s_k | x_k, q_k) = \rho_\lambda x_k^{\frac{\lambda+\eta}{\eta}} \tilde{x}_k^{-\frac{(\lambda+2\eta)}{\eta}} \exp\left(\frac{x_k}{\eta \tilde{x}_k} - \frac{\tilde{x}_k s_k}{\eta} \right) \delta\left(q_k - x_k^{-1}\right). \tag{4.13}$$

Here ρ_λ is a normalisation constant, whose value will be fixed later. Knowing R_λ we can construct the Q-operator, which is defined as the trace of the product of $\hat{R}_\lambda^{(k)}$ as follows (Ghose Choudhury and Roy Chowdhury, 2000):

$$\hat{Q}_\lambda = tr \prod_{k=1}^{N} \hat{R}^{(k)}_{\lambda - c_k} \tag{4.14}$$

where we have introduced additional inhomogeneity parameters $\{c_k\}_{k=1}^N$ at lattice sites. With the aid of (4.13) we may define the kernel of the Q-operator by

$$K_\lambda \equiv \int dq_N \ldots \int dq_1 \prod_{k=1}^{N} R_{\lambda - c_k}(\tilde{x}_k, q_{k-1}|x_k, q_k) \tag{4.15}$$

where we have imposed the condition $s_k = q_{k-1}$. Using (4.13) and writing $\mu_k = \frac{(\lambda - c_k + 2\eta)}{\eta}$ we may express the kernel of (4.15) by

$$K_\lambda = \prod_{k=1}^{N} \rho_{\lambda - c_k} x_k^{\mu_k - 1} \tilde{x}_k^{-\mu_k} \exp\left(\frac{x_k}{\eta \tilde{x}_k} - \frac{\tilde{x}_k}{\eta x_{k-1}}\right) \tag{4.16}$$

whence upon taking the logarithm of Eq. (4.16) one obtains

$$\begin{aligned} \log K_\lambda &= \sum_{k=1}^{N} \log\left[\rho_{\lambda - c_k} x_k^{\mu_k - 1} \tilde{x}_k^{-\mu_k} \exp\left(\frac{x_k}{\eta \tilde{x}_k} - \frac{\tilde{x}_k}{\eta x_{k-1}}\right)\right] \\ &= \frac{1}{\eta} \sum_{k=1}^{N} \left[(\lambda - c_k)\log\left(\frac{x_k}{\tilde{x}_k}\right) + \frac{x_k}{\tilde{x}_k} - \frac{\tilde{x}_k}{x_{k-1}}\right] + \sum_k \tilde{\Delta}_k \end{aligned} \tag{4.17}$$

where

$$\tilde{\Delta}_k = -\mu_k \log \eta + \log \frac{\Gamma(\mu_k)}{2i \sin(\pi \mu_k)} \tag{4.18}$$

is a quantum correction. The appearance of $\Gamma(\mu_k)$ will be apparent from what follows. But it is evident from (4.17) that the exponential of the classical generating function $F(\lambda)$ for canonical Bäcklund transformation as given by (3.12) is related to the kernel K_λ and represents therefore the semi classical limit. From equations (4.1) and (4.13) we observe that for any suitable function $\phi(q, x)$

$$\begin{aligned} \hat{R}_{\lambda - c} : \phi(x, q) &\longmapsto \int dx \int dq\, R_{\lambda - c}(\tilde{x}, s|x, q)\phi(x, q) \\ &= \int dx \int dq \rho_{\lambda - c} x^{\frac{\lambda - c + \eta}{\eta}} \tilde{x}^{-\frac{(\lambda - c + 2\eta)}{\eta}} \exp\left(\frac{x}{\eta \tilde{x}} - \frac{\tilde{x} s}{\eta}\right) \delta(q - x^{-1})\phi(x, q) \end{aligned} \tag{4.19}$$

Consequently if $\phi(q, x) = 1$ then one can write (4.19) as

$$\hat{R}_{\lambda-c} : 1 \longmapsto \rho_{\lambda-c} \int dx \left(\frac{x}{\eta\tilde{x}}\right)^{\frac{\lambda-c+\eta}{\eta}} \eta^{\frac{\lambda-c+\eta}{\eta}} \tilde{x}^{-1} \exp\left(\frac{x}{\eta\tilde{x}} - \frac{\tilde{x}s}{\eta}\right). \tag{4.20}$$

Introducing $\sigma = -\frac{x}{\eta\tilde{x}}$ and $z = -\frac{\tilde{x}s}{\eta}$ allows us to express Eq. (4.20) as

$$R_{\lambda-c} : 1 \longmapsto -\rho_{\lambda-c}\eta^{\frac{\lambda-c+2\eta}{\eta}} e^{z} \int (-\sigma)^{\frac{\lambda-c+2\eta}{\eta}-1} e^{-\sigma} d\sigma. \tag{4.21}$$

Next from the integral representation of Euler's Gamma function (Roy Chowdhury and Ghose Choudhury, 2004) viz.

$$\Gamma(\mu) = -\frac{1}{2i\,\sin(\pi\mu)} \int_c (-\sigma)^{\mu-1} e^{-\sigma} d\sigma. \tag{4.22}$$

where the contour C is depicted in figure [1]. We may express (4.21) in the form

$$\hat{R}_{\lambda-c} : 1 \longmapsto \rho_{\lambda-c}\eta^{\mu} e^{z} 2i\,\sin(\pi\mu)\Gamma(\mu) \tag{4.23}$$

where $\mu = (\lambda - c + 2\eta)/\eta$, so that we may fix the normalisation factor $\rho_{\lambda-c}$ by letting

$$\rho_{\lambda-c} = \frac{\eta^{-\mu}}{2i\,\sin(\pi\mu)\Gamma(\mu)} \tag{4.24}$$

which enables us to write

$$\hat{R}_{\lambda-c} : 1 \longmapsto e^{\frac{-\tilde{x}s}{\eta}}$$

with

$$R_{\lambda-c}(\tilde{x}, s|x, q) = \frac{\eta^{-\mu}}{2i\,\sin(\pi\mu)\Gamma(\mu)} x^{\mu-1}\tilde{x}^{-\mu} \exp\left(\frac{x}{\eta\tilde{x}} - \frac{\tilde{x}s}{\eta}\right) \delta(q - x^{-1}). \tag{4.25}$$

The expression for the quantum correction given by Eq. (4.18) is evident from the above form of $R_{\lambda-c}$ as given in Eq. (4.25).

We shall next consider how the spectrum of the system may be analysed.

5. ANALYTICAL BETHE ANSATZ

As stated earlier in section one, derivation of the spectrum of the model constitutes an important aspect of any quantum mechanical analysis of an integrable system. To this end we consider the quantum mechanical version of the Lax operator in Eq. (2.1) in which the discrete dynamical variables (x_n, p_n) are considered to as non commuting quantum mechanical variables obeying appropriate commutation relations as given by Eq. (4.2)

In the quantum regime the Lax operator satisfies the following algebra, which serves as the defining relation for the quantum R-matrix:

$$R(u-v)l_k(u)\otimes l_k(v) = l_k(v)l_k(u)R(u-v). \tag{5.1}$$

For the model under consideration the quantum R-matrix is of the form

$$R(u-v) = \begin{pmatrix} u-v-i\eta & 0 & 0 & 0 \\ 0 & -i\eta & u-v & 0 \\ 0 & u-v & -i\eta & 0 \\ 0 & 0 & 0 & u-v-i\eta \end{pmatrix}. \tag{5.2}$$

The corresponding monodromy matrix is defined by

$$T_N(u) = l_N(u)l_{N-1}(u)\ldots l_1(u) \tag{5.3}$$

and the transfer matrix is obtained from the trace of $T_N(u)$ i.e.

$$t(u) \equiv trT_N(u) = tr\prod_{k=1}^{N} l_k(u). \tag{5.4}$$

It is well known that Eq. (5.1) then implies,

$$R(u-v)T_N(u)\otimes T_N(v) = T_N(v)\otimes T_N(u)R(u-v) \tag{5.5}$$

which may be regarded as representing a kind of global version of the local relation (5.1). From Eq. (5.3) it is evident that $T_N(u)$ is a 2×2 matrix with operator entries, so that formally we may express $T_N(u)$ in the form

$$T_N(u) = \begin{pmatrix} A_N(u) & B_N(u) \\ C_N(u) & D_N(u) \end{pmatrix} \tag{5.6}$$

so that

$$t(u) = A_N(u) + D_N(u). \tag{5.7}$$

From (5.5) it is a staright forward matter to derive the following commutation rules for the entries of $T_N(u)$:

$$[A_N(u), A_N(v)] = [B_N(u), B_N(v)] = [C_N(u), C_N(v)] = [D_N(u), D_N(v)] = 0 \tag{5.8}$$

$$[A_N(u), C_N(v)] = \frac{i\eta}{u-v}[C_N(u)A_N(v) - C_N(v)A_N(u)] \tag{5.9}$$

$$[D_N(u), C_N(v)] = \frac{i\eta}{u-v}[D_N(u)C_N(v) - D_N(v)C_N(u)] \tag{5.10}$$

a close look at the Lax operator considered in our case however reveals an interesting feature, namely the non- existence of a pseudovacuum state— a feature also shared by the well known Toda lattice. This prevents us from being able to apply the techniqueof Algebraic Bethe Ansatz for deducing the spectrum of the model. It is necessary therefore to take recourse to the so called analytical Bethe ansatz which relies on the explicit construction of suitable polynomial expansions of the elements of the monodromy matrix from its defining Eq. (5.3) (Whittaker and Watson, 1958; Sklyanin, 1992). In the present case one finds by explicit multiplication that

$$\begin{aligned} A_N(u) &= O(u^{N-2}) \\ C_N(u) &= x_1(u^{N-1} + \ldots) \\ D_N(u) &= u_N + i\left(\sum_K^N p_k x_k\right) u^{N-1} + \ldots \end{aligned} \tag{5.11}$$

Now let us denote by $\hat{u}_\alpha$ $(\alpha = 1, 2, 3 \ldots, N-1)$ the zeros of $C_N(u)$ so that

$$C_N(\hat{u}_\alpha) = 0, \quad \alpha = 1, 2 \ldots N-1. \tag{5.12}$$

This allows us to redefine the operator $C_N(u)$ of (5.11) in the following form, in terms of its zeros,

$$C_N(u) = \hat{x}_1 \prod_{\alpha=1}^{N-1} (u - \hat{u}_\alpha). \tag{5.13}$$

Next we define the operator $\hat{v}_\alpha^\pm$ by left substitution $u \Longrightarrow \hat{u}_\alpha$ into the operators $A_N(u)$ and $D_N(u)$ i.e.,

$$\hat{v}_\alpha^+ \equiv A_N(u \Longrightarrow \hat{u}_\alpha), \quad \hat{v}_\alpha^- \equiv D_N(u \Longrightarrow \hat{u}_\alpha), \ \ \alpha = 1, 2, 3 \ldots, N-1 \tag{5.14}$$

The zeros of $C_N(u)$, i.e., $\{\hat{u}_\alpha\}_{\alpha=1}^{N-1}$ together with $\{\hat{v}_\alpha^\pm\}_{\alpha=1}^{N-1}$ may be shown to be the separation variables of the eigen value problem. However note that the zeros of $C_N(u)$ provide us with only (N-1) "co-ordinates." Hence it is necessary to identify another pair of conjugate variables, as we are dealing with N lattice sites. We take the following

$$\hat{u}_N \equiv \hat{x}_1 \quad \text{and} \quad \hat{v}_N \equiv \prod_{k=1}^{N} \hat{p}_k \hat{x}_k \tag{5.15}$$

The commutation rules (5.8)–(5.10) then allow us to derive the following commutation relations:

$$\begin{aligned} &[\hat{u}_N, \hat{u}_\alpha] = [\hat{u}_N, \hat{v}_\alpha^\pm] = [\hat{v}_N, \hat{u}_\alpha] = [\hat{v}_N, \hat{v}_\alpha^\pm] = 0, \\ &\alpha = 1, 2 \ldots N-1 \end{aligned} \tag{5.16}$$

$$[\hat{u}_N, \hat{v}_N] = i\eta\hat{u}_N, \quad [\hat{v}^{\pm}_{\alpha}, \hat{u}_{\beta}] = \mp i\eta\hat{v}^{\pm}_{\alpha}\delta_{\alpha\beta} \tag{5.17}$$

$$[\hat{v}^{+}_{\alpha}, \hat{v}^{+}_{\beta}] = 0 = [\hat{v}^{-}_{\alpha}, \hat{v}^{-}_{\beta}]. \tag{5.18}$$

In view of the above, we may employ Lagrange interpolation formulae to rewrite the elements of the monodromy matrix in the following manner.

$$C_N(u) = \hat{u}_N \prod_{\alpha=1}^{N-1}(u - \hat{u}_{\alpha}) \tag{5.19}$$

$$A_N(u) = \sum_{\alpha=1}^{N-1}\prod_{\beta\neq\alpha}^{N-1}\frac{u-\hat{u}_{\beta}}{\hat{u}_{\alpha}-\hat{u}_{\beta}}\hat{v}^{+}_{\alpha} \tag{5.20}$$

$$D_N(u) = \left(u + i\hat{v}_N + \sum_{\alpha=1}^{N-1}\hat{u}_{\alpha}\right)\prod_{\alpha=1}^{N-1}(u-\hat{u}_{\alpha}) + \sum_{\alpha=1}^{N-1}\prod_{\beta\neq\alpha}^{N-1}\frac{u-\hat{u}_{\beta}}{\hat{u}_{\alpha}-\hat{u}_{\beta}}\hat{v}^{-}_{\alpha}. \tag{5.21}$$

Let us now consider the action of the transfer matrix on any symmetric function $\phi(\hat{u}_1, \hat{u}_2, \ldots \hat{u}_{N-1})$:

$$\begin{aligned} t(u)\phi(\hat{u}_1, \hat{u}_2, \ldots \hat{u}_{N-1}) &= \hat{A}_N(u)\phi(\hat{u}_1, \hat{u}_2, \ldots \hat{u}_{N-1}) + \hat{D}_N(u)\phi(\hat{u}_1, \hat{u}_2, \ldots \hat{u}_{N-1}) \\ &= \sum_{\alpha=1}^{N-1}\prod_{\beta\neq\alpha}^{N-1}\frac{u-\hat{u}_{\beta}}{\hat{u}_{\alpha}-\hat{u}_{\beta}}\hat{v}^{+}_{\alpha}\phi(\hat{u}_1, \hat{u}_2, \ldots \hat{u}_{N-1}) \\ &\quad + \left(u + i\hat{v}_N + \sum_{\alpha=1}^{N-1}\hat{u}_{\alpha}\right)\prod_{\alpha=1}^{N-1}(u-\hat{u}_{\alpha})\phi(\hat{u}_1, \hat{u}_2, \ldots \hat{u}_{N-1}) \\ &\quad + \sum_{\alpha=1}^{N-1}\prod_{\beta\neq\alpha}^{N-1}\frac{u-\hat{u}_{\beta}}{\hat{u}_{\alpha}-\hat{u}_{\beta}}\hat{v}^{-}_{\alpha}\phi(\hat{u}_1, \hat{u}_2, \ldots \hat{u}_{N-1}). \end{aligned} \tag{5.22}$$

Then the substitution $u \longrightarrow \hat{u}_{\alpha}$ gives

$$t(\hat{u}_{\alpha})\phi(\hat{u}_1, \hat{u}_2, \ldots \hat{u}_{N-1}) = \hat{v}^{+}_{\alpha}\phi(\hat{u}_1, \hat{u}_2, \ldots \hat{u}_{N-1}) + \hat{v}^{-}_{\alpha}\phi(\hat{u}_1, \hat{u}_2, \ldots \hat{u}_{N-1}) \tag{5.23}$$

The actions of $\hat{v}^{\pm}_{\alpha}$ on $\phi(\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_{N-1})$ can be understood from the commutation relations (5.17) Hence

$$\begin{aligned} &[\hat{v}^{\pm}_{\alpha}, \hat{u}_{\beta}] = \mp i\eta\hat{v}^{\pm}_{\alpha}\delta_{\alpha\beta} \\ &\Rightarrow \hat{v}^{\pm}_{\alpha}\hat{u}_{\alpha} = (\hat{u}_{\alpha} \mp i\eta)\hat{v}^{\pm}_{\alpha} \end{aligned}$$

it follows that

$$\hat{v}^{\pm}_{\alpha}\phi(\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_{N-1}) = \phi(\hat{u}_1, \hat{u}_2, \ldots \hat{u}_{\alpha} \mp i\eta, \ldots \hat{u}_{N-1})\hat{v}^{\pm}_{\alpha}. \tag{5.24}$$

Hence Eq. (5.23) now assumes the form

$$t(\hat{u}_\alpha)\phi(\hat{u}_1, \hat{u}_2, \ldots \hat{u}_{N-1}) = \phi(\hat{u}_1, \hat{u}_2, \ldots \hat{u}_\alpha - i\eta, \ldots \hat{u}_{N-1}) + \phi(\hat{u}_1, \hat{u}_2, \ldots \hat{u}_\alpha + i\eta, \ldots \hat{u}_{N-1}) \tag{5.25}$$

If we now assume that $\phi(\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_{N-1})$ is separable so that

$$\phi(\hat{u}_1, \hat{u}_2, \ldots \hat{u}_{N-1}) + \prod_{\alpha=1}^{N-1} \psi(\hat{u}_\alpha) \tag{5.26}$$

we find that

$$t(\hat{u}_\alpha)\prod_{\alpha=1}^{N-1} \psi(\hat{u}_\alpha) = \{\psi(\hat{u}_1)\ldots\psi(\hat{u}_\alpha - i\eta)\ldots\psi(\hat{u}_{N-1})\} + \{\psi(\hat{u}_1)\ldots\psi(\hat{u}_\alpha + i\eta)\ldots\psi(\hat{u}_{N-1})\}$$

which evidently leads to the following relation, namely

$$t(u)\psi(u) = \psi(u - i\eta) + \psi(u + i\eta). \tag{5.27}$$

Assuming further that $\psi(u) = \prod_j (u - u_j)$ then leads to the conclusion that

$$t(u)\prod_j (u - u_j) = \prod_j (u - i\eta - u_j) + \prod_j (u + i\eta - u_j). \tag{5.28}$$

The zeros u_j of $\psi(u)$ may be determined by setting $u = u_k$ in (5.28) which leads to the following set of equations determining them:

$$\prod_{j\neq k} \frac{u_k - u_j - i\eta}{u_k - u_j + i\eta} = 1. \tag{5.29}$$

6. DISCUSSION

In this communication we have obtained a new hierar chy of discrete nonlinear equations from the Lax operator (2.1). The explicit form of the equations for the simplest case are given in (2.8). The system is observed to posses a standard classical r- matrix structure, from which we have obtained an explicit form of classical Bäcklund transformation for the system under consideration. It is important to note that by the very manner of its construction, the Bäcklund transformation so obtained is also a canonical transformation being derivable from a generating function whose explicit form is given in (3.12). We have then analysed the corresponding quantum version of such a canonical Bäcklund transformation, which gives rise to the notion of a Q operator. The representation of later being in the form of an integral operator whose kernel and normalisation factor have being explicitly determined. Finally we have outlined the manner in which the spectrum of the quantum mechanical version of the Lax operator may be deduced. It is nteresting to note that because the Lax operator does not have a pseudo vacuum state, we have had to formulate the quantum inverse problem in terms of the Analytical Bethe Ansatz rather than the usual algebraic procedure.

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